Available online at www.sciencedirect.com
ScienceDirect
Knowledge-Based Systems 21 (2008) 164–171
Knowledge-Based SYSTEMS
www.elsevier.com/locate/knosys

On multi-period multi-attribute decision making

Zeshui Xu

Department of Management Science and Engineering, School of Economics and Management, Tsinghua University, Beijing 100084, China

Received 15 December 2006; received in revised form 16 May 2007; accepted 22 May 2007
Available online 26 May 2007

Abstract

Multiple attribute decision making (MADM) is an important part of modern decision science. It has been extensively applied to various areas such as society, economics, military, management, etc., and has been receiving more and more attention over the last decades. To date, however, most research has focused on single-period multi-attribute decision making in which all the original decision information is given at the same period, and a number of methods have been proposed to solve this kind of problems. This paper is devoted to investigating the multi-period multi-attribute decision making (MP-MADM) problems where the decision information (including attribute weights and attribute values) are provided by decision maker(s) at different periods. We define the concept of dynamic weighted averaging (DWA) operator, and introduce some methods, such as the arithmetic series based method, geometric series based method and normal distribution based method, to obtain the weights associated with the DWA operator. Based on the DWA operator, we develop an approach to MP-MADM. Moreover, we extend the DWA operator and the developed approach to solve the MP-MADM problems where all the attribute values provided at different periods are expressed in interval numbers, and use a possibility-degree formula to rank and select the given alternatives.

Keywords: Multi-period multi-attribute decision making (MP-MADM); Dynamic weighted averaging (DWA) operator; Arithmetic series based method; Geometric series based method; Normal distribution based method

1. Introduction

Multiple attribute decision making (MADM) involves finding the most desirable alternative(s) from a discrete set of feasible alternatives with respect to a finite set of attributes. MADM has been being a hot research topic over the last decades, and has been extensively applied to various areas such as society, economics, military, management, etc. [1–10]. A lot of studies have been done on single-period MADM, for example, Saaty [1] used 1–9 ratio scale to compare each pair of attributes (alternatives) so as to construct a multiplicative preference relation, from which the eigenvalue method is used to derive the attribute (alternative) weights, and finally he aggregated these weights of attributes and alternatives by using the weighted averaging operator to get the ranking of alternatives. Hwang and Yoon [2] introduced a technique for order performance by similarity to ideal solution (TOPSIS), one of the known classical method for MADM. The fundamental idea of the TOPSIS is that the chosen alternative should have the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. In the process of TOPSIS, the attribute values and the attribute weights are given as exact numerical values. Chen [11] extended the TOPSIS to the fuzzy environment in which the rating of each alternative and the weight of each attribute are described by linguistic terms which can be expressed in triangular fuzzy numbers. A vertex procedure was developed to calculate the distance between two triangular fuzzy numbers, and a closeness coefficient was defined to determine the ranking of all alternatives by calculating the distances to both the fuzzy positive-ideal solution and fuzzy negative-ideal solution simultaneously. Park and Kim [12] presented the characteristic of weak dominance and proposed dominance graph, and also presented

E-mail address: xu_zeshui@263.net

0950-7051/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.knosys.2007.05.007

an algorithm, by using a separable linear program technique based on linear-(in)equality-styled information, to derive the most preferable alternative(s). Kim et al. [13] presented an interactive procedure for multiple attribute group decision making problems with incomplete information. The procedure needs each group member to express his/her preference in relation to an additive value model with incomplete preference statements, and calculates group's utility range based on each group member's incomplete information. They also described theoretic models for establishing group's pairwise dominance relations with group's utility ranges by using a separable linear programming technique. Ma [14] et al. utilized the subjective information provided by a decision maker and the objective information to establish a two-objective programming model to determine the attribute weights, and then utilized the simple additive weighting method to rank alternatives. Xu and Chen [15] developed an interactive method for solving multiple attribute group decision making problems under fuzzy environments. The method can be used in situations where the information about attribute weights is partly known and the decision maker weights are expressed in exact numerical values or triangular fuzzy numbers, and the attribute values take the form of triangular fuzzy numbers. Chen and Hwang [4] proposed a method for fuzzy MADM problems. The approach first converts the fuzzy decision data (linguistic terms or fuzzy numbers) into crisp scores, and then applies an appropriate classical MADM method (such as TOPSIS, etc.) to determine the ranking of alternatives. It is capable of solving large size real-world problems which contain a mixture of fuzzy and crisp data. Brans and Vincke [16] presented a preference ranking organization method for enrichment evaluations (PROMETHEE I) to build a partial ranking among alternatives, and proposed an enhanced method, PROMETHEE II, which allows ranking all alternatives, including incomparable ones, to obtain a total classification. Pawlak et al. [17–19] applied rough set theory to multi-attribute decision analysis and developed some rough set based approaches to solving the MADM problems. Xu and co-workers [20,21] developed two evidential reasoning approaches to multi-attribute decision analysis under uncertainty and interval uncertainty, respectively. Yager [22] introduced the concept of ordered weighted averaging (OWA) operator, whose fundamental characteristic is the re-ordering step, in particular, an argument is not associated with a particular weight, but rather a weight is associated with a particular ordered position of the arguments. Then, he used the OWA operator to aggregate the decision information expressed in exact numerical values in MADM. Bordogna et al. [23] established a linguistic model based on linguistic OWA operators to evaluate the consensual judgment and consensus degree for each alternative in multiple attribute group decision making, where both the decision makers' evaluations of the alternatives and the degree of consensus are expressed linguistically. Xu [24] proposed the uncertain linguistic ordered weighted averaging (ULOWA) operator and uncertain linguistic hybrid aggregation (ULHA) operator, and then based on these two operators, developed an approach to multiple attribute group decision making with uncertain linguistic information. Fenton and Wang [25] investigated the MADM problems taking account of uncertainty, risk and confidence. They used linguistic variables and triangular fuzzy numbers to model the decision maker's risk and confidence attitudes in order to define a more complete MCDM solution, and employed a practical travel problem to assess the developed MCDM technique. This technique is useful for tackling imprecision and subjectivity in complex, ill-defined and human-oriented decision problems. Xu [26] explored the MADM problems with linguistic information, in which the information about attribute weights is incompletely known, and the attribute values take the form of linguistic variables. Xu introduced some approaches to obtaining the weight information of attributes, and then established an optimization model based on the ideal point of attribute values, by which the attribute weights can be determined. Furthermore, Xu utilized the numerical weighting linguistic average (NWLA) operator to aggregate the linguistic variables corresponding to each alternative and ranked the alternatives by means of the aggregated linguistic information, and then applied the developed method to the ranking and selection of propulsion/manoeuvring system of a double-ended passenger ferry.

However, in many real-life situations, such as multi-period (multi-stage) investment decision making, medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation, etc., the original argument information may be collected from different periods. Thus, it is an interesting and important research issue. In this paper, we shall focus on this issue and organize this paper in six sections. Section 2 defines the concept of dynamic weighted averaging (DWA) operator, and introduces some methods to determine the weights associated with the DWA operator. Based on the DWA operator, Section 3 develops an approach to solving the MP-MADM problems where all the attribute values at different periods are expressed in exact numerical values. Section 4 extends the DWA operator and the developed approach to solve the MP-MADM problems where all the attribute values provided at different periods are expressed in interval numbers. Section 5 gives an illustrative example, and Section 6 ends the paper.

2. Dynamic weighted averaging (DWA) operator

Information aggregation is a key step in the process of MADM. At present, many aggregation operators have been developed to aggregate argument information [27]. As was pointed out in [28], current research on aggregation operators mainly focuses on time independent operators. Accordingly, as time is not taken into account, operators and weights are usually kept constant. However, in many real-life situations, the argument information may be provided at different periods (stages). Thus, it is necessary to investigate the time dependent aggregation operators. In what follows, we define a dynamic weighted averaging (DWA) operator.

Definition 1. Let $a(t_1), a(t_2), \ldots, a(t_p)$ be a collection of real-valued arguments collected from p different periods t_k $(k = 1, 2, \ldots, p)$, and $\lambda(t) = (\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p))^{\mathrm{T}}$ be the weight vector of the periods t_k $(k = 1, 2, \ldots, p)$, then

$$DWA_{\lambda(t)}(a(t_1), a(t_2), \ldots, a(t_p)) = \sum_{k=1}^{p} \lambda(t_k) a(t_k) \tag{1}$$

is called a dynamic weighted averaging (DWA) operator, where

$$\lambda(t_k) \geqslant 0, \quad k = 1, 2, \ldots, p, \quad \sum_{k=1}^{p} \lambda(t_k) = 1 \tag{2}$$

Clearly, one important step of the DWA operator is to determine the weight vector $\lambda(t) = (\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p))^{\mathrm{T}}$ of the periods t_k $(k = 1, 2, \ldots, p)$. In general, $\lambda(t)$ can be given by decision maker(s) directly, or can be obtained by using one of the following methods:

(1) *Arithmetic series based method:* Suppose that the difference value between the weight $\lambda(t_{k+1})$ and its adjacent weight $\lambda(t_k)$ is α, for each k, i.e.,

$$\lambda(t_{k+1}) - \lambda(t_k) = \alpha, \quad k = 1, 2, \ldots, p-1 \tag{3}$$

In this case, we have

$$\lambda(t_k) = \eta + (k-1)\alpha, \quad \eta + (k-1)\alpha \geqslant 0 \tag{4}$$

with the condition (2). From (4) we have

(i) If $\alpha = 0$, then $\eta = 1/n$, i.e., $\lambda(t_k) = 1/n$, $k = 1, 2, \ldots, p$, which indicates that all the weights $\lambda(t_k)(k = 1, 2, \ldots, p)$ are equal.
(ii) If $\alpha > 0$, then $\lambda(t_k) < \lambda(t_{k+1})$, $k = 1, 2, \ldots, p-1$, which indicates that the larger k, the greater $\lambda(t_k)$.
(iii) If $\alpha < 0$, then $\lambda(t_k) > \lambda(t_{k+1})$, $k = 1, 2, \ldots, p-1$, which indicates that the larger k, the smaller $\lambda(t_k)$.

(2) *Geometric series based method*: Suppose that the weight $\lambda(t_{k+1})$ is β times as good as it adjacent weight $\lambda(t_k)$, for each k i.e.,

$$\lambda(t_{k+1}) = \beta\lambda(t_k), \quad \beta > 0, \ k = 1, 2, \ldots, p-1 \tag{5}$$

In this case, we have

$$\lambda(t_k) = \eta\beta^{k-1}, \quad \eta, \beta > 0, \ k = 1, 2, \ldots, p-1 \tag{6}$$

Then by (2), it follows that

$$\eta = \frac{1}{\sum_{k=1}^{p} \beta^{k-1}}, \quad \beta > 0 \tag{7}$$

and thus

$$\lambda(t_k) = \frac{\beta^{k-1}}{\sum_{j=1}^{p} \beta^{j-1}}, \quad \beta > 0, \ k = 1, 2, \ldots, p \tag{8}$$

From (8), we have

(i) If $\beta = 1$, then $\eta = 1/n$, i.e., $\lambda(t$,5.65.3,(i) gch indica allF-18 (

Obviously, the normal distribution based method assigns the largest weights to the medial period, and the further the period t_k departs from the medial period, the smaller the weight assigned to the period t_k.

Based on the DWA operator and the weight generation methods above, in the next section, we shall develop a straightforward approach to solving MP-MADM problems.

3. An approach to MP-MADM

The MP-MADM problem under study can be described as follows:

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of n feasible alternatives, and let $U = \{u_1, u_2, \ldots, u_m\}$ be a finite set of attributes. Suppose that there are p periods t_k $(k = 1, 2, \ldots, p)$, whose weight vector is $\lambda(t) = (\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p))^{\mathrm{T}}$, where $\lambda(t_k) \geqslant 0$, $k = 1, 2, \ldots, p$, $\sum_{k=1}^{p} \lambda(t_k) = 1$, and $w(t_k) = (w_1(t_k), w_2(t_k), \ldots, w_m(t_k))^{\mathrm{T}}$ is the weight vector of the attributes $u_i (i = 1, 2, \ldots, m)$ at the period t_k, where $w_i(t_k) \geqslant 0$, $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} w_i(t_k) = 1$. Let $A(t_k) = (a_{ij}(t_k))_{m \times n}$ be a decision matrix (see Table 1), where $a_{ij}(t_k)$ is an attribute value, which takes the form of positive real numbers, of the alternative $x_j \in X$ with respect to the attribute $u_i \in U$ at the period t_k.

Consider that there are generally benefit attributes and cost attributes in the MP-MADM problems. In order to measure all attributes in dimensionless units and to facilitate inter-attribute comparisons, in what follows, we normalize each decision matrix $A(t_k) = (a_{ij}(t_k))_{m \times n}$ into a corresponding decision matrix $R(t_k) = (r_{ij}(t_k))_{m \times n}$, by using the following formulas:

$$r_{ij}(t_k) = \frac{a_{ij}(t_k)}{\max_j\{a_{ij}(t_k)\}} \quad \text{for benefit attribute } u_i, \qquad (16)$$
$$i = 1, 2, \ldots, m, \;\; j = 1, 2, \ldots, n, \;\; k = 1, 2, \ldots, p$$

$$r_{ij}(t_k) = \frac{\min\limits_j\{a_{ij}(t_k)\}}{a_{ij}(t_k)} \quad \text{for cost attribute } u_i, \qquad (17)$$
$$i = 1, 2, \ldots, m, \;\; j = 1, 2, \ldots, n, \;\; k = 1, 2, \ldots, p$$

Now we first utilize the weighted averaging operator

$$r_j(t_k) = \sum_{i=1}^{m} w_i(t_k) r_{ij}(t_k), \quad j = 1, 2, \ldots, n \qquad (18)$$

to aggregate the attribute values $r_{ij}(t_k)$ $(i = 1, 2, \ldots, m)$ in the ith column of the normalized decision matrix $R(t_k)$ into an overall attribute value $r_j(t_k)$ of the alternative x_j at the period t_k.

Then we utilize the DWA operator (1) to aggregate the overall attribute values $r_j(t_k)$ $(k = 1, 2, \ldots, p)$ of the p different periods t_k $(k = 1, 2, \ldots, p)$ into a complex overall attribute value r_j of the alternative x_j, where

$$r_j = \sum_{k=1}^{p} \lambda(t_k) r_j(t_k), \quad j = 1, 2, \ldots, n \qquad (19)$$

Therefore, we can rank all the alternatives x_j $(j = 1, 2, \ldots, n)$ and then select the most desirable one(s) in accordance with the values of r_j $(j = 1, 2, \ldots, n)$.

Consider that, sometimes, the input arguments may not be specified, but value ranges can be obtained, that is, each input argument is given in the form of interval values rather than exact numerical values, in the next section, we shall extend the above approach to MP-MADM in which all the attribute values are expressed in interval numbers.

4. An approach to MP-MADM under interval uncertainty

We first define an uncertain dynamic weighted averaging (UDWA) operator as below:

Definition 2. Let $\tilde{a}(t_1), \tilde{a}(t_2), \ldots, \tilde{a}(t_p)$ be a collection of interval-valued arguments collected from the p different periods $t_k (k = 1, 2, \ldots, p)$, where $\tilde{a}(t_k) = [\tilde{a}^L(t_k), \tilde{a}^U(t_k)]$, $\tilde{a}^L(t_k)$ and $\tilde{a}^U(t_k)$ are the lower and upper limits of $\tilde{a}(t_k)$, respectively, $k = 1, 2, \ldots, p$, $\lambda(t) = (\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p))^{\mathrm{T}}$ be the weight vector of the periods $t_k (k = 1, 2, \ldots, p)$, then

$$UDWA_{\lambda(t)}(\tilde{a}(t_1), \tilde{a}(t_2), \ldots, \tilde{a}(t_p)) = \sum_{k=1}^{p} \lambda(t_k) \tilde{a}(t_k)$$
$$= \left[\sum_{k=1}^{p} \lambda(t_k) \tilde{a}^L(t_k), \sum_{k=1}^{p} \lambda(t_k) \tilde{a}^U(t_k) \right] \qquad (20)$$

is called an uncertain dynamic weighted averaging (UDWA) operator, where the weight vector $\lambda(t)$ satisfies the condition (2) and can be determined by using one of the methods introduced in Section 2.

In what follows, we develop an approach to MP-MADM under interval uncertainty:

Step 1. For a MP-MADM problem, X, U, $\lambda(t)$, and $w(t_k)$ $(k = 1, 2, \ldots, p)$ are as defined in Section 3. Let $\tilde{A}(t_k) = (\tilde{a}_{ij}(t_k))_{m \times n}$ be a decision matrix (see Table 2), where $\tilde{a}_{ij}(t_k) = [\tilde{a}_{ij}^L(t_k), \tilde{a}_{ij}^U(t_k)]$ is an attribute value, which

Table 1
Decision matrix $A(t_k)$

	x_1	x_2	$\cdots$	x_n
u_1	$a_{11}(t_k)$	$a_{12}(t_k)$	$\cdots$	$a_{1n}(t_k)$
u_2	$a_{21}(t_k)$	$a_{22}(t_k)$	$\cdots$	$a_{2n}(t_k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m	$a_{m1}(t_k)$	$a_{m2}(t_k)$	$\cdots$	$a_{mn}(t_k)$

Table 2
Decision matrix $\tilde{A}(t_k)$

	x_1	x_2	$\cdots$	x_n
u_1	$[\tilde{a}_{11}^L(t_k), \tilde{a}_{11}^U(t_k)]$	$[\tilde{a}_{12}^L(t_k), \tilde{a}_{12}^U(t_k)]$	$\cdots$	$[\tilde{a}_{1n}^L(t_k), \tilde{a}_{1n}^U(t_k)]$
u_2	$[\tilde{a}_{21}^L(t_k), \tilde{a}_{21}^U(t_k)]$	$[\tilde{a}_{22}^L(t_k), \tilde{a}_{22}^U(t_k)]$	$\cdots$	$[\tilde{a}_{2n}^L(t_k), \tilde{a}_{2n}^U(t_k)]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m	$[\tilde{a}_{m1}^L(t_k), \tilde{a}_{m1}^U(t_k)]$	$[\tilde{a}_{m2}^L(t_k), \tilde{a}_{m2}^U(t_k)]$	$\cdots$	$[\tilde{a}_{mn}^L(t_k), \tilde{a}_{mn}^U(t_k)]$

takes the form of interval numbers, of the alternative $x_j \in X$ with respect to the attribute $u_i \in U$ at the period t_k.

Step 2. Normalize each decision matrix $\tilde{A}(t_k) = (\tilde{a}_{ij}(t_k))_{m\times n}$ into a corresponding decision matrix $\tilde{R}(t_k) = (\tilde{r}_{ij}(t_k))_{m\times n}$, by using the following formulas:

$$\tilde{r}_{ij}(t_k) = \tilde{a}_{ij}(t_k)/\sum_{j=1}^{n}\tilde{a}_{ij}(t_k) \quad \text{for benefit attribute } u_i,$$

$$i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n,\ \ k = 1,2,\ldots,p \tag{21}$$

$$\tilde{r}_{ij}(t_k) = \left(1/\tilde{a}_{ij}(t_k)\right)/\sum_{j=1}^{n}\left(1/\tilde{a}_{ij}(t_k)\right) \quad \text{for cost attribute } u_i,$$

$$i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n,\ \ k = 1,2,\ldots,p \tag{22}$$

By the operations of interval numbers, we rewrite (21) and (22) as (23) and (24), respectively:

$$\begin{cases} \tilde{r}^L_{ij}(t_k) = \tilde{a}^L_{ij}(t_k)/\sum_{j=1}^{n}\tilde{a}^U_{ij}(t_k) \\ \tilde{r}^U_{ij}(t_k) = \tilde{a}^U_{ij}(t_k)/\sum_{j=1}^{n}\tilde{a}^L_{ij}(t_k) \end{cases} \quad \text{for benefit attribute } u_i,$$

$$i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n,\ \ k = 1,2,\ldots,p \tag{23}$$

$$\begin{cases} \tilde{r}^L_{ij}(t_k) = (1/\tilde{a}^U_{ij}(t_k))/\sum_{j=1}^{n}(1/\tilde{a}^L_{ij}(t_k)) \\ \tilde{r}^U_{ij}(t_k) = (1/\tilde{a}^L_{ij}(t_k))/\sum_{j=1}^{n}(1/\tilde{a}^U_{ij}(t_k)) \end{cases} \quad \text{for cost attribute } u_i,$$

$$i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n,\ \ k = 1,2,\ldots,p \tag{24}$$

where $\tilde{r}_{ij}(t_k) = [\tilde{r}^L_{ij}(t_k), \tilde{r}^U_{ij}(t_k)]$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $k = 1,2,\ldots,p$.

Step 3. Utilize the uncertain weighted averaging operator

$$\tilde{r}_j(t_k) = \left[\tilde{r}^L_j(t_k), \tilde{r}^U_j(t_k)\right] = \sum_{i=1}^{m} w_i(t_k)\tilde{r}_{ij}(t_k) = \left[\sum_{i=1}^{m} w_i(t_k)\tilde{r}^L_{ij}(t_k), \sum_{i=1}^{m} w_i(t_k)\tilde{r}^U_{ij}(t_k)\right], \quad j = 1, 2, \ldots, n \tag{25}$$

to aggregate the attribute values $\tilde{r}_{ij}(t_k)$ $(i = 1,2,\ldots,m)$ in the ith column of the normalized decision matrix $\tilde{R}(t_k)$ into an overall attribute value $\tilde{r}_j(t_k)$ of the alternative x_j at the period t_k.

Step 4. Utilize the UDWA operator (20) to aggregate the overall attribute values $\tilde{r}_j(t_k)$ $(k = 1,2,\ldots,p)$ of the p different periods t_k $(k = 1,2,\ldots,p)$ into a complex overall attribute value $\tilde{r}_j$ of the alternative x_j, where

$$\tilde{r}_j = \left[\tilde{r}^L_j, \tilde{r}^U_j\right] = \sum_{k=1}^{p} \lambda(t_k)\tilde{r}_j(t_k) = \left[\sum_{k=1}^{p} \lambda(t_k)\tilde{r}^L_j(t_k), \sum_{k=1}^{p} \lambda(t_k)\tilde{r}^U_j(t_k)\right],$$

$$j = 1,2,\ldots,n \tag{26}$$

Step 5. To rank the interval arguments $\tilde{r}_j$ $(j = 1,2\cdot,\ldots,n)$, we first compare each $\tilde{r}_i$ with all $\tilde{r}_j$ $(j = 1,2,\ldots,n)$ by using the following possibility-degree formula [29]:

$$p(\tilde{r}_i \geqslant \tilde{r}_j) = \max\left\{1 - \max\left(\frac{\tilde{r}^U_j - \tilde{r}^L_i}{\tilde{r}^U_i - \tilde{r}^L_i + \tilde{r}^U_j - \tilde{r}^L_j}, 0\right), 0\right\} \tag{27}$$

For convenience, we let $p_{ij} = p(\tilde{r}_i \geqslant \tilde{r}_j)$, and then construct a complementary matrix $P = (p_{ij})_{n\times n}$, which satisfies [29,30].

$$p_{ij} \geqslant 0, \quad p_{ij} + p_{ji} = 1, \quad p_{ii} = \frac{1}{2}, \ \ i,j = 1,2,\ldots,n \tag{28}$$

Step 6. Summing all elements in each line of matrix P, we have

$$p_i = \sum_{j=1}^{n} p_{ij}, \quad i = 1,2,\ldots,n \tag{29}$$

Then we can reorder the interval arguments $\tilde{r}_j$ $(j = 1,2,\ldots,n)$ in descending order in accordance with the values of p_j $(j = 1,2,\ldots,n)$.

Step 7. Rank all the alternatives x_j $(j = 1,2,\ldots,n)$ and then select the most desirable one(s) in accordance with the values of r_j $(j = 1,2,\ldots,n)$.

5. Illustrative case

In this section, we utilize a practical MP-MADM problem to illustrate the application of the developed approaches.

An investment company wants to invest a sum of money in the best option. There is a panel with five possible companies x_j $(j = 1,2,\ldots,5)$ in which to invest the money: (1) x_1 is a car company; (2) x_2 is a food company; (3) x_3 is a computer company; (4) x_4 is an arms company; and (5) x_5 is a TV company. The attributes which are considered here in selection of the five possible companies are: (1) u_1 is economical benefit; (2) u_2 is social benefit; and (3) u_3 is the environmental pollution, where the attributes u_1 and u_2 are benefit attributes, and the attribute u_3 is cost attribute. The investment company uses 0–1 scale to evaluate the performance of the companies x_j $(j = 1,2,\ldots,5)$ in 2004–2006 according to the attributes u_i $(i = 1,2,3)$, and constructs, respectively, the decision matrices $A(t_k)$ $(k = 1,2,3$, here, t_1 denotes "2004", t_2 denotes "2005", and t_3 denotes "2006") as listed in Tables 3–5. Let $\lambda(t) = (1/6, 2/6, 3/6)^{\mathrm{T}}$ be the weight vector of the years t_k $(k = 1,2,3)$, and let $w(t_1) = (0.40, 0.40, 0.20)^{\mathrm{T}}$, $w(t_2) = (0.40, 0.35, 0.25)^{\mathrm{T}}$, and $w(t_3) = (0.40, 0.30, 0.30)^{\mathrm{T}}$ be the weight vectors of the attributes u_i $(i = 1,2,3)$ in the years t_k $(k = 1,2,3)$, respectively.

We first normalize the decision matrices $A(t_k)$ $(k = 1,2,3)$ into the corresponding decision matrices $R(t_k)$ $(k = 1,2,3)$ (see Tables 6–8) by using the formulas (16)

Table 3
Decision matrix $A(t_1)$

	x_1	x_2	x_3	x_4	x_5
u_1	0.75	0.95	0.80	0.90	0.85
u_2	0.85	0.70	0.90	0.80	0.85
u_3	0.50	0.45	0.35	0.40	0.55

Table 4
Decision matrix $A(t_2)$

	x_1	x_2	x_3	x_4	x_5
u_1	0.80	0.90	0.85	0.85	0.90
u_2	0.90	0.85	0.80	0.75	0.90
u_3	0.45	0.40	0.40	0.50	0.60

Table 5
Decision matrix $A(t_3)$

	x_1	x_2	x_3	x_4	x_5
u_1	0.90	0.85	0.95	0.90	0.95
u_2	0.85	0.90	0.85	0.80	0.85
u_3	0.30	0.35	0.45	0.45	0.50

Table 6
Normalized decision matrix $R(t_1)$

	x_1	x_2	x_3	x_4	x_5
u_1	0.7895	1.0000	0.8421	0.9474	0.8947
u_2	0.9444	0.7778	1.0000	0.8889	0.9444
u_3	0.7000	0.7778	1.0000	0.8750	0.6364

Table 7
Normalized decision matrix $R(t_2)$

	x_1	x_2	x_3	x_4	x_5
u_1	0.8889	1.0000	0.9444	0.9444	1.0000
u_2	1.0000	0.9444	0.8889	0.8333	1.0000
u_3	0.8889	1.0000	1.0000	0.8000	0.6667

Table 8
Normalized decision matrix $R(t_3)$

	x_1	x_2	x_3	x_4	x_5
u_1	0.9474	0.8947	1.0000	0.9474	1.0000
u_2	0.9444	1.0000	0.9444	0.8889	0.9444
u_3	1.0000	0.8571	0.6667	0.6667	0.6000

and (17) and then utilize (18) to aggregate the attribute values in the ith column of $R(t_k)$ into an overall attribute value $r_j(t_k)$ of the alternative x_j at the year t_k:

$$r_1(t_1) = 0.8333, \quad r_2(t_1) = 0.8667, \quad r_3(t_1) = 0.9368,$$
$$r_4(t_1) = 0.9095, \quad r_5(t_1) = 0.8629, \quad r_1(t_2) = 0.9278,$$
$$r_2(t_2) = 0.9805, \quad r_3(t_2) = 0.9389, \quad r_4(t_2) = 0.8694,$$
$$r_5(t_2) = 0.9167, \quad r_1(t_3) = 0.9623, \quad r_2(t_3) = 0.9150,$$
$$r_3(t_3) = 0.8833, \quad r_4(t_3) = 0.8456, \quad r_5(t_3) = 0.8633$$

By using (1), we aggregate the overall attribute values $r_j(t_k)$ $(k = 1,2,3)$ of the years t_k $(k = 1,2,3)$ into a complex overall attribute value r_j of the alternative x_j

$$r_1 = 0.9293, \quad r_2 = 0.9288, \quad r_3 = 0.9107,$$
$$r_4 = 0.9642, \quad r_5 = 0.8810$$

Therefore, we can rank all the alternatives x_j $(j = 1,2,\ldots,5)$ in accordance with the values of r_j $(j = 1,2,\ldots,5)$

$$x_4 \succ x_1 \succ x_2 \succ x_3 \succ x_5$$

and thus the best alternative (company) is x_4.

Sometimes, however, the evaluation information can not be provided with exact numerical value, but value ranges can be obtained due to the increasing complexity and uncertainty of real-life decision making problems. In this case, we reconsider the above MP-MADM problem as follows:

The investment company uses 0–1 scale to evaluate the performance of the companies x_j $(j = 1,2,\ldots,5)$ in 2004–2006 according to the attributes u_i $(i = 1,2,3)$, and constructs, respectively, the decision matrices $\tilde{A}(t_k)$ $(k = 1,2,3)$, in which all the attribute values are expressed in interval numbers (see Tables 9–11):

To get the best alternatives, the following steps are involved:

Step 1. Normalize the decision matrices $\tilde{A}(t_k)$ $(k = 1,2,3)$ into the corresponding decision matrices $\tilde{R}(t_k)(k = 1, 2, 3)$ (see Tables 12–14), by using the formulas (23) and (24).

Step 2. Utilize (25) to aggregate the attribute values in the ith column of the normalized decision matrix $\tilde{R}(t_k)$

Table 9
Decision matrix $\tilde{A}(t_1)$

	x_1	x_2	x_3	x_4	x_5
u_1	[0.70, 0.80]	[0.90, 0.95]	[0.80, 0.90]	[0.90, 1.0]	[0.80, 0.85]
u_2	[0.85, 0.90]	[0.70, 0.75]	[0.85, 0.90]	[0.80, 0.90]	[0.75, 0.80]
u_3	[0.30, 0.50]	[0.40, 0.50]	[0.30, 0.40]	[0.20, 0.30]	[0.50, 0.60]

Table 10
Decision matrix $\tilde{A}(t_2)$

	x_1	x_2	x_3	x_4	x_5
u_1	[0.80, 0.85]	[0.90, 0.95]	[0.80, 0.90]	[0.85, 0.95]	[0.85, 0.90]
u_2	[0.90, 0.95]	[0.80, 0.85]	[0.70, 0.80]	[0.80, 0.85]	[0.80, 0.90]
u_3	[0.35, 0.45]	[0.35, 0.40]	[0.40, 0.45]	[0.30, 0.50]	[0.55, 0.65]

Table 11
Decision matrix $\tilde{A}(t_3)$

	x_1	x_2	x_3	x_4	x_5
u_1	[0.90, 0.95]	[0.85, 0.90]	[0.85, 0.95]	[0.90, 0.95]	[0.80, 0.95]
u_2	[0.85, 0.95]	[0.90, 1.0]	[0.75, 0.85]	[0.80, 0.90]	[0.80, 0.85]
u_3	[0.30, 0.35]	[0.30, 0.40]	[0.45, 0.50]	[0.35, 0.45]	[0.45, 0.50]

Table 12
Normalized decision matrix $\tilde{R}(t_1)$

	x_1	x_2	x_3	x_4	x_5
u_1	[0.1556,0.1951]	[0.2000,0.2317]	[0.1778,0.2195]	[0.2222,0.2439]	[0.1778,0.2073]
u_2	[0.1977,0.2278]	[0.1628,0.1899]	[0.1977,0.2278]	[0.1860,0.2278]	[0.1744,0.2152]
u_3	[0.1237,0.2899]	[0.1237,0.2174]	[0.1546,0.2899]	[0.2062,0.4348]	[0.1031,0.1739]

Table 13
Normalized decision matrix $\tilde{R}(t_2)$

	x_1	x_2	x_3	x_4	x_5
u_1	[0.1758,0.2024]	[0.1978,0.2262]	[0.1758,0.2143]	[0.1868,0.2262]	[0.1868,0.2143]
u_2	[0.2069,0.2375]	[0.1839,0.2125]	[0.1609,0.2000]	[0.1839,0.2125]	[0.1839,0.2250]
u_3	[0.1663,0.2726]	[0.1870,0.2726]	[0.1663,0.2385]	[0.1496,0.3180]	[0.1151,0.1734]

Table 14
Normalized decision matrix $\tilde{R}(t_3)$

	x_1	x_2	x_3	x_4	x_5
u_1	[0.1915,0.2209]	[0.1809,0.2093]	[0.1809,0.2209]	[0.1915,0.2209]	[0.1702,0.2209]
u_2	[0.1868,0.2317]	[0.1978,0.2439]	[0.1648,0.2073]	[0.1758,0.2195]	[0.1758,0.2073]
u_3	[0.2045,0.2879]	[0.1790,0.2879]	[0.1432,0.1919]	[0.1591,0.2467]	[0.1432,0.1919]

into an overall attribute value $\tilde{r}_j(t_k)$ of the alternative x_j at the year t_k:

$$\tilde{r}_1(t_1) = [0.1661, 0.2271], \quad \tilde{r}_2(t_1) = [0.1699, 0.2121],$$
$$\tilde{r}_3(t_1) = [0.1811, 0.2369], \quad \tilde{r}_4(t_1) = [0.2045, 0.2756],$$
$$\tilde{r}_5(t_1) = [0.1615, 0.2038], \quad \tilde{r}_1(t_2) = [0.1843, 0.2322],$$
$$\tilde{r}_2(t_2) = [0.1902, 0.2330], \quad \tilde{r}_3(t_2) = [0.1682, 0.2153],$$
$$\tilde{r}_4(t_2) = [0.1765, 0.2444], \quad \tilde{r}_5(t_2) = [0.1679, 0.2078],$$
$$\tilde{r}_1(t_3) = [0.1940, 0.2442], \quad \tilde{r}_2(t_3) = [0.1854, 0.2433],$$
$$\tilde{r}_3(t_3) = [0.1648, 0.2081], \quad \tilde{r}_4(t_3) = [0.1771, 0.2282],$$
$$\tilde{r}_5(t_3) = [0.1638, 0.2081]$$

Step 3. Utilize the UDWA operator (20) to aggregate the overall attribute values $\tilde{r}_j(t_k)(k = 1, 2, 3)$ of the years $t_k(k = 1, 2, 3)$ into a complex overall attribute value $\tilde{r}_j$ of the alternative x_j:

$$\tilde{r}_1 = [0.1861, 0.2374], \quad \tilde{r}_2 = [0.1844, 0.2347],$$
$$\tilde{r}_3 = [0.1686, 0.2153]$$
$$\tilde{r}_4 = [0.1815, 0.2415], \quad \tilde{r}_5 = [0.1648, 0.2073]$$

Step 4. Compare each $\tilde{r}_i$ with all $\tilde{r}_j$ $(j = 1, 2, \ldots, 5)$ by using (27), and then construct a complementary matrix:

$$P = \begin{bmatrix} 0.5 & 0.5217 & 0.7020 & 0.5022 & 0.7740 \\ 0.4783 & 0.5 & 0.6814 & 0.4823 & 0.7532 \\ 0.2980 & 0.3186 & 0.5 & 0.3168 & 0.5661 \\ 0.4978 & 0.5177 & 0.6832 & 0.5 & 0.7483 \\ 0.2260 & 0.2468 & 0.4339 & 0.2517 & 0.5 \end{bmatrix}$$

Step 5. Summing all elements in each line of matrix P, we have

$$p_1 = 0.2250, \quad p_2 = 0.2198, \quad p_3 = 0.1750, \quad p_4 = 0.2223, \quad p_5 = 0.1579$$

Then we can reorder the interval arguments $\tilde{r}_j$ $(j = 1, 2, \ldots, 5)$ in descending order in accordance with the values of p_j $(j = 1, 2, \ldots, 5)$

$$\tilde{r}_4 > \tilde{r}_1 > \tilde{r}_2 > \tilde{r}_3 > \tilde{r}_5$$

and thus we get the ranking of alternatives

$$x_4 \succ x_1 \succ x_2 \succ x_3 \succ x_5$$

therefore, the best alternative (company) is x_4.

6. Conclusions

In this paper, we have introduced two dynamic aggregation operators, i.e., the dynamic weighted averaging (DWA) operator and uncertain dynamic weighted averaging (UDWA) operator. Both the operators take time into account in the aggregation process, and thus are time independent operators, which can overcome the disadvantages of the existing static aggregation operators (or called time independent operators). To determine the weights associated with these two dynamic aggregation operators, we have proposed three simple weight generation methods including the arithmetic series based method, geometric series based method, and normal distribution based method, all of which can sufficiently embody the characteristics of the provided original arguments. The DWA operator can be used to aggregate the real-valued arguments, and the UDWA operator can be used to aggregate the interval-valued arguments. All these real-valued or interval-valued arguments are obtained from different periods.

Furthermore, we have developed a DWA operator based approach to multi-period multi-attribute decision making (MP-MADM) where all the decision information is expressed in real-valued arguments, and developed an UDWA operator based approach to MP-MADM under interval uncertainty, in which all the attribute values take the form of interval numbers. Both the approaches have been detailedly verified with a practical case.

The further research may focus on the application of the developed approaches to the fields of medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation, etc.

Acknowledgements

The author is very grateful to the editor, Professor Ernest Edmonds, and the anonymous referees for their insightful and constructive comments and suggestions that have led to an improved version of this paper. The work was supported by the National Natural Science Founda-